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DESIGN OF TWO-DIMENSIONAL IIR DIGITAL FILTERS VIA LINEAR PROGRAMMING

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Abstract. A general method for the design of the magnitude response of 2-D IIR digital filters by using linear programming is presented. The formulation of the linear problem is achieved by developing appropriate linear constraints which ensure the minimization of a realistic approximation error of the magnitude response. Moreover, the stability of the resultant filter is guaranteed and the computational cost is decreased by exploiting the symmetries appearing in the linear programming problem. The method leads to satisfactory designs as shown with two examples.

Zusammenfassung. Vorgestellt wird ein allgemeine Vorgehensweise für die Approximation des Frequenzgangs Zweidimensionaler rekursiver Digitalfilter mit Hilfe linearer Programmierung. Hierbei wird die Problemstellung für die lineare Programmierung derart formuliert, daß geeignete lineare Einschränkungen entwickelt werden, die die Minimierung eines realistischen Fehlermaßes für die Approximation des Frequenzgangs gewährleisten. Darüber hinaus wird die Stabilität des resultierenden Filters garantiert. Der Rechenaufwand wird dadurch erniedrigt, daß die Symmetrien ausgenutzt werden, die im Laufe der linearen Programmierungsaufgabe auftreten. Die Methode führt zu zufriedenstellenden Ergebnissen, wie anhand zweier Beispiele gezeigt wird.

Résumé. Une méthode générale pour l'élaboration de la réponse d'amplitude d'un filtre RII numérique bidimensionnel en utilisant la programmation linéaire est présentée. La formulation du problème linéaire est obtenue en développant des contraintes linéaires appropriées garantissant la minimisation d'une erreur realiste d'approximation de la reponse d'amplitude. En plus, la stabilité du filtre résultant est garantie et le coût de calcul est diminué en exploitant les symétries apparaissant dans le problème de programmation linéaire. La méthode conduit à des élaborations satisfaisantes ainsi qu'il est montré par deux examples.

Keywords. Two-dimensional digital filters, infinite impulse response (IIR) filters, design of filters, linear programming, stability of 2-D digital filters.

1. Introduction

The first efforts for the design of 2-D IIR digital filters were based on the transformations applied on 1-D digital filters [4, 9, 12]. However, transformations from the 1-D case do not generally provide satisfactory approximations for 2-D rational functions. To this end, over the past few years many computer-aided optimization approaches have been developed for the design of 2-D IIR digital filters [6, 7, 10, 13].

The design problem of IIR digital filters may be viewed as a nonlinear complex approximation problem. For its solution, various approaches have been proposed based on the design of 1-D IIR digital filter; however, they involve some difficulties and disadvantages which are discussed in the sequel. Specifically, the lack of the fundamental theorem of algebra in the 2-D case leads to problems of ensuring stability since it is reduced to the test of a 2-D polynomial, whose eigenvalues are actually irreducible algebraic curves.

Nonlinear optimization methods used for the solution of the nonlinear problem are associated with well-known disadvantages such as lack of convergence of the algorithm to the global optimal solution and considerable computation and programming costs. Another disadvantage of nonlinear optimization techniques refers to the choice of an initial starting point which is usually required to lie close to the extremum.

For the successful resolution of these difficulties, linear models have been developed which approximate the 2-D IIR digital filter design problem. These models are solved using the classical linear programming (LP) algorithms [8]. It is obvious that a linear approximation to an inherently nonlinear problem presents difficulties in formulating the optimization procedure correctly, but they are counterbalanced by the many advantages of LP. Linear formulations developed thus far are extensions of techniques originally used for the design of 1-D IIR digital filters [2, 18] and are based upon the work of Matthews et al. [11]. They can be distinguished into two major categories.

The first linear approximation approach involves the design of 2-D IIR digital filters via approximation, in a weighted Chebyshev (minimax) sense, of the filter magnitude-squared specifications. Dudgeon's [5] work belongs to this category. His method is based upon a differential correction algorithm [1] and is solved using an LP algorithm recursively. This approach suffers from a few serious drawbacks of which the most important are the high computation cost and the procedural complexity of determining the actual filter transfer function from the magnitude-squared expression via discrete Hilbert transform and minimum phase stability criteria [17]. A new formulation based on the Chebyshev approximation and requiring the solution of an LP problem only once with a consequent drastic decrease in computer cost was recently proposed by the present authors [14].

The second category of linear approximation problems allows for the design of 2-D IIR filters on the basis of not only the magnitude response but also linear phase specifications. Concurrently, it is possible to use linear stability constraints for filter stability as proposed by Chottera and Jullien [3]. One serious drawback is attributed to this technique: solution of the design problem does not always guarantee a good filter since the objective function to be minimized is but a weighted error.

The main goal of the present work is to appropriately modify existing formulations so that solution of the approximation problem results in acceptable magnitude response characteristics. This is accomplished by the development of appropriate linear constraints capable of assuring a good approximation error for the magnitude response while guaranteeing filter stability. Actually, the optimization problem is not a minimax problem any more but has been properly formulated in order to satisfactorily design the magnitude response by minimizing a linear cost function. Moreover, the symmetries contained in the linear formulation are exploited so that memory requirements and computer costs are kept to a minimum.

Two examples illustrating the proposed method are presented. In the first, a comparison is attempted between the new approach and the method suggested by Chottera and Jullien [3]. In the second, a low-pass filter is designed with octagonal and therefore approximately circular symmetry characteristics.

2. Design approach

An approximation to the design of a 2-D transfer function, as proposed in [3], may be based on a variation of the complex Chebyshev approximation, originally introduced for the 1-D case by Matthews et al. [11].

In the sequel, a more general formulation of the problem than that of Chottera and Jullien is presented first, thus relaxing the design requirements. Specifically, let $\bar{H}(z_{1m}, z_{2n})$ be the transfer function of a two-dimensional recursive digital filter where, by the overbar, the complex nature of $\bar{H}(z_1, z_2)$ is denoted. At each sampling point (m, n), belonging to the sampling region S, $\bar{H}(z_{1m}, z_{2m})$ may be expressed in the form

$$\bar{H}(z_{1m}, z_{2n}) = \bar{A}(z_{1m}, z_{2n})/\bar{B}(z_{1m}, z_{2n}) = \left(\sum_{i=0}^{m_1}\sum_{j=0}^{n_1} a_{ij} z_{1m}^i z_{2n}^j\right) \Bigg/ \left(\sum_{i=0}^{m_2}\sum_{j=0}^{n_2} b_{ij} z_{1m}^i z_{2n}^j\right) \tag{1}$$

for all $m = 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$ and $(m, n) \in S$, where $b_{00} = 1$ and $z_{1m} = e^{-jw_{1m}}$, $z_{2n} = e^{-jw_{2n}}$ are the normalized frequencies in the frequency domain. Let also $\bar{G}_{mn} = \bar{G}(z_{1m}, z_{2n})$ be the complex function containing the filter's magnitude and phase specifications. Define now the complex error function by the following formula:

$$\bar{E}_{mn} = \bar{G}_{mn} - \bar{H}(z_{1m}, z_{2n}). \tag{2}$$

The $\bar{E}_{mn}$ are known small variables and represent an admissible complex approximation error at every sampling point (m, n). Note that this error is introduced in order to relax the requirements imposed by the specifications. From (2), by equating real and imaginary parts, the following equations result:

$$[\mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})]\,\mathrm{Re}(\bar{B}(z_{1m}, z_{2n})) - [\mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})]\,\mathrm{Im}(\bar{B}(z_{1m}, z_{2n})) - \mathrm{Re}(\bar{A}(z_{1m}, z_{2n})) = 0, \tag{3}$$

$$[\mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})]\,\mathrm{Re}(\bar{B}(z_{1m}, z_{2n})) + [\mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})]\,\mathrm{Im}(\bar{B}(z_{1m}, z_{2n})) - \mathrm{Im}(\bar{A}(z_{1m}, z_{2n})) = 0 \tag{4}$$

for all $(m, n) \in S$, where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary part of $(\cdot)$, respectively.

After some algebraic manipulations similar to those reported in [3], the LP problem is formulated as follows:

Minimize ξ

Subject to

$$\left| \sum_{\substack{i=0\\i+j\neq 0}}^{m_2}\sum_{j=0}^{n_2} (C_{ij})_{mn} b_{ij} - \sum_{i=0}^{m_1}\sum_{j=0}^{n_1} \mathrm{Re}(z_{1m}^i z_{2n}^j) a_{ij} + \mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn}) \right| \leq \xi, \tag{5}$$

$$\left| \sum_{\substack{i=0\\i+j\neq 0}}^{m_2}\sum_{j=0}^{n_2} (D_{ij})_{mn} b_{ij} - \sum_{i=0}^{m_1}\sum_{j=0}^{n_1} \mathrm{Im}(z_{1m}^i z_{2n}^j) a_{ij} + \mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn}) \right| \leq \xi,$$

$\xi \geq 0$, for all $(m, n) \in S$, where

$$(C_{ij})_{mn} = \mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Re}(z_{1m}^i z_{2n}^j) - \mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Im}(z_{1m}^i z_{2n}^j), \tag{6}$$

$$(D_{ij})_{mn} = \mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Re}(z_{1m}^i z_{2n}^j) + \mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Im}(z_{1m}^i z_{2n}^j). \tag{7}$$

From the above formulation, it is seen that the desired approximation has the form

$$|\bar{H}(z_{1m}, z_{2n})| \to |\bar{G}_{mn}| \tag{8}$$

and

$$\measuredangle \bar{H}(z_{1m}, z_{2n}) \to \measuredangle \bar{G}_{mn}. \tag{9}$$

The linear phase specifications are given by a relation of the form

$$\measuredangle \bar{G}_{mn} = -(T_1 w_{1m} + T_2 w_{2n}), \tag{10}$$

where T_1, T_2 are positive numbers. The above conditions are satisfied when $|\bar{E}_{mn}|$ is a small positive number. If $|\bar{E}_{mn}| = 0$, we obtain the formulation introduced by Chottera and Jullien [3].

The design problem considered here is aimed at simultaneously satisfying both magnitude and linear phase specifications. The objective function ξ, as defined above, only represents a weighted error and does not suffice for a satisfactory design. To this end, a new approximation error, δ_{mn}, of the magnitude response, at each sampling point, is introduced; this error influences the design of the magnitude response only and is defined as follows:

$$\delta_{mn} = |\bar{G}_{mn} - \bar{E}_{mn}|^2 - |\bar{H}(z_{1m}, z_{2n})|^2. \tag{11}$$

Let $(\xi_R)_{mn}$, $(\xi_I)_{mn}$ denote the values of the expressions at the left-hand side of (3) and (4) respectively. Note that

$$\xi = \max_{(m,n)\in S} \{|\xi_R|_{mn}, |\xi_I|_{mn}\} \tag{12}$$

as can be seen from (3), (4), and (5).

Now, equations (3) and (4) take the form

$$(\xi_R)_{mn} + \mathrm{Re}(\bar{A}(z_{1m}, z_{2n})) = \mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Re}(\bar{B}(z_{1m}, z_{2n})) - \mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Im}(\bar{B}(z_{1m}, z_{2n})), \tag{13a}$$

$$(\xi_I)_{mn} + \mathrm{Im}(\bar{A}(z_{1m}, z_{2n})) = \mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Re}(\bar{B}(z_{1m}, z_{2n})) + \mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})\,\mathrm{Im}(\bar{B}(z_{1m}, z_{2n})). \tag{13b}$$

If we take the sum of the squares of (13a) and (13b), we arrive at the following expression:

$$|\bar{G}_{mn} - \bar{E}_{mn}|^2 |\bar{B}(z_{1m}, z_{2n})|^2 = |\bar{A}(z_{1m}, z_{2n})|^2 + 2(\xi_R)_{mn}\,\mathrm{Re}(\bar{A}(z_{1m}, z_{2n})) + 2(\xi_I)_{mn}\,\mathrm{Im}(\bar{A}(z_{1m}, z_{2n})) + (\xi_R)^2_{mn} + (\xi_I)^2_{mn}. \tag{14}$$

Substituting $\mathrm{Re}(\bar{A}(z_{1m}, z_{2n}))$ and $\mathrm{Im}(\bar{A}(z_{1m}, z_{2n}))$ from equations (13) in (14), the error δ_{mn} may be written in the form

$$\delta_{mn} = [2\,\mathrm{Re}(\bar{B}(z_{1m}, z_{2n}))[\mathrm{Re}(\bar{G}_{mn} - E_{mn})(\xi_R)_{mn} + \mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})(\xi_I)_{mn}] + 2\,\mathrm{Im}(\bar{B}(z_{1m}, z_{2n})) \times [\mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})(\xi_I)_{mn} - \mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})(\xi_R)_{mn}] - (\xi_R)^2_{mn} - (\xi_I)^2_{mn}]/|\bar{B}(z_{1m}, z_{2n})|^2, \tag{15}$$

which expresses δ_{mn} as a nonlinear function of the unknown coefficients of the transfer function.

It is evident from (15) that minimization of $(\xi_R)_{mn}$, $(\xi_I)_{mn}$ (and therefore of ξ) does not guarantee the minimization of $|\delta_{mn}|$, which represents a realistic measure of the approximation error of the magnitude response, since the denominator $\bar{B}(z_{1m}, z_{2n})$ is involved in the expression.

This implies that formulation (5) indeed does not lead to a satisfactory approximation. Therefore, the minimization of $|\delta_{mn}|$, $(m, n) \in S$, must be ensured. Sufficient conditions for the minimization of $|\delta_{mn}|$ result from (15), by taking into account the inequality $a^2 + b^2 \geq 2|a|\,|b|$, as follows:

$$|\delta_{mn}| \leq |\mathrm{Re}(\bar{G}_{mn} - \bar{E}_{mn})| \left[\frac{|(\xi_I)_{mn}|}{|\mathrm{Re}(\bar{B}(z_{1m}, z_{2n}))|} + \frac{|(\xi_R)_{mn}|}{|\mathrm{Im}(\bar{B}(z_{1m}, z_{2n}))|}\right] + |\mathrm{Im}(\bar{G}_{mn} - \bar{E}_{mn})| \left[\frac{|(\xi_I)_{mn}|}{|\mathrm{Im}(\bar{B}(z_{1m}, z_{2n}))|} + \frac{|(\xi_R)_{mn}|}{|\mathrm{Re}(\bar{B}(z_{1m}, z_{2n}))|}\right] - \frac{(\xi_R)^2_{mn} + (\xi_I)^2_{mn}}{|\bar{B}(z_{1m}, z_{2n})|^2}. \tag{16}$$

Since $|\mathrm{Re}(\bar{G}_{mn}-\bar{E}_{mn})|\leq 1$ and $|\mathrm{Im}(\bar{G}_{mn}-\bar{E}_{mn})|\leq 1$, it is clear that the constraints

$$\frac{\xi}{|\mathrm{Re}(\bar{B}(z_{1m},z_{2n}))|}\leq\delta_1,\qquad \frac{\xi}{|\mathrm{Im}(\bar{B}(z_{1m},z_{2n}))|}\leq\delta_2 \tag{17a, b}$$

are appropriate and sufficient conditions, with δ_1, δ_2 small positive numbers, leading to a small value for $|\delta_{mn}|$. Condition (17a) is equivalent to the following two constraints:

$$\mathrm{Re}(\bar{B}(z_{1m},z_{2n}))\geq\frac{\xi}{\delta_1},\qquad \mathrm{Re}(\bar{B}(z_{1m},z_{2n}))\leq\frac{-\xi}{\delta_1}. \tag{18a, b}$$

Moreover, Chottera and Jullien [3] stipulate that suitable linear conditions for stability may be written as

$$\mathrm{Re}(\bar{B}(z_{1m},z_{2n}))\geq\varepsilon\quad\text{for } |z_{1m}|=1 \text{ and } |z_{2n}|=1. \tag{19}$$

We observe that (18a) and (19) have the same form since ξ and δ_1 are positive numbers. To this end, we choose constraint (18a) in place of (17a) since it satisfies stability constraint (19). In conclusion, conditions (17b) and (18a) are the additional constraints to be included in the formulation of linear problem (5), in order to ensure the minimization of $|\delta_{mn}|$. However, constraints (17b) and (18a) are not linear with respect to the coefficients ξ, δ_1, and δ_2. To resolve this problem, the following variable transformation is introduced:

$$\begin{aligned}
&\xi'=1/\xi,\qquad \delta_1'=1/\delta_1,\qquad \delta_2'=1/\delta_2,\\
&\alpha_{ij}=a_{ij}/\xi,\quad i=1,2,\ldots,m_1,\ j=1,2,\ldots,n_1,\\
&\beta_{ij}=b_{ij}/\xi,\quad i=1,2,\ldots,m_2,\ j=1,2,\ldots,n_2.
\end{aligned} \tag{20}$$

From the above it can be seen that it is preferable to minimize δ_1, δ_2 rather than ξ. Thus, problem (5) takes the form

$$\begin{aligned}
&\text{Maximize } \delta_1'+Q\delta_2'\\
&\text{Subject to}\\
&\left|\sum_{\substack{i=0\\ i+j\neq 0}}^{m_2}\sum_{j=0}^{n_2}(C_{ij})_{mn}\beta_{ij}-\sum_{i=0}^{m_1}\sum_{j=0}^{n_1}\mathrm{Re}(z_{1m}^i z_{2n}^j)\alpha_{ij}+\mathrm{Re}(\bar{G}_{mn}-\bar{E}_{mn})\xi'\right|\leq 1,\\
&\left|\sum_{\substack{i=0\\ i+j\neq 0}}^{m_2}\sum_{j=0}^{n_2}(D_{ij})_{mn}\beta_{ij}-\sum_{i=0}^{m_1}\sum_{j=0}^{n_1}\mathrm{Im}(z_{1m}^i z_{2n}^j)\alpha_{ij}+\mathrm{Im}(\bar{G}_{mn}-\bar{E}_{mn})\xi'\right|\leq 1,\\
&\delta_1'-\xi'-\sum_{\substack{i=0\\ i+j\neq 0}}^{m_2}\sum_{j=0}^{n_2}\mathrm{Re}(z_{1m}^i z_{2n}^j)\beta_{ij}\leq 0,\qquad \delta_2'-\left|\sum_{i=0}^{m_2}\sum_{j=0}^{n_2}\mathrm{Im}(z_{1m}^i z_{2n}^j)\beta_{ij}\right|\leq 0,
\end{aligned} \tag{21}$$

where ξ', δ_1', $\delta_2'\geq 0$ and Q is an appropriate positive constant.

Linear problem (21) contains a large number of constraints. This fact hinders an easy solution of the primal problem necessitating a formulation in its dual form. The linear constraints of (21) also contain many symmetries; this is due to the fact that the coefficient matrix in the revised Simplex algorithm contains some symmetries with respect to its elements. Taking advantage of these symmetries reduces memory requirements while speeding up the solution. The solution of the dual problem is achieved via a revised Simplex method appropriately modified so that block symmetries in the coefficient matrix are fully utilized. The algorithm was implemented on a PDP 11-34 computer using double precision arithmetic.

Example 1. The design of a 2-D IIR digital filter is considered using the proposed formulation and that of Chottera and Jullien [3] for purposes of comparison. Let the transfer function of the filter be specified in the form

$$\begin{aligned} H(z_1, z_2) = {} & [a_1(1+z_1^2+z_2^2+z_1^2z_2^2)^2+a_2(z_1+z_2+z_1^2z_2+z_1z_2^2)^2+a_3z_1^2z_2^2+a_4(1+z_1^2+z_2^2+z_1^2z_2^2) \\ & \times(z_1+z_2+z_1^2z_2+z_1z_2^2)+a_5(1+z_1^2+z_2^2+z_1^2z_2^2)z_1z_2+a_6z_1z_2(z_1+z_2+z_1^2z_2+z_1z_2^2)] \\ & \times[1+b_12(z_1+z_2)+2b_2(z_1^2+z_2^2)+2b_3(z_1+z_2)(z_1z_2+z_1^2+z_2^2)+b_4(4z_1z_2+z_1^2+z_2^2) \\ & +2b_5(2z_1z_2+z_1^4+z_2^4+2z_1^2z_2^2)+4b_6z_1z_2(z_1^2+z_2^2+z_1z_2)+2b_7(z_1+z_2)z_1z_2 \\ & +(z_1^2+z_2^2+z_1z_2)+2b_8z_1z_2(z_1+z_2)+2b_9z_1^2z_2^2(z_1^2+z_2^2)+2b_{10}z_1^2z_2^2(z_1+z_2) \\ & +2b_{11}z_1^3z_2^3(z_1+z_2)+b_{12}z_1^2z_2^2+b_{13}z_1^4z_2^4+2b_{14}z_1^3z_2^3]^{-1}. \end{aligned} \tag{22}$$

The above transfer function has properly been chosen (with linearized coefficient terms) to possess quadrantal symmetry. Thus, the characteristics of quadrantal symmetry (which approximates circular symmetry) [15, 16] are exhibited while the total number of coefficients, relative to the general form (1), is reduced. Moreover, let the desired filter's magnitude specifications be of the form

$$\operatorname{Re}(\bar{G}_{mn}) = \begin{cases} G_1 & \text{for } 0 \leq w_{mn} < R_p, \\ \dfrac{G_1-G_2}{R_p-R_s} w_{mn} + \dfrac{R_pG_2-R_sG_1}{R_p-R_s} & \text{for } R_p \leq W_{mn} < R_s, \\ G_2 & \text{for } R_s \leq w_{mn} \leq \sqrt{2}\,\pi, \end{cases} \tag{23}$$

where $w_{mn}=\sqrt{w_{1m}^2+w_{2n}^2}$ and $G_1=1$, $G_2=0$, $|\bar{E}_{mn}|=0$, $R_p=0.1\pi$ and $R_s=0.3\pi$.

The total number of sampling points is 148 (56, 28, and 64 in the pass-band, transition-band and stop-band case, respectively). At first, we consider $T_1=T_2=6$ and the design problem is solved using the Chottera and Jullien method. The resultant coefficients are given in Table 1, while ξ takes the value of 0.004049502.

The above solution leads to an unsatisfactory design for the magnitude response. This is seen from the fact that the commonly used l_2 criterion

$$J = \sum\sum_{(m,n)\in S} [|\bar{H}_{mn}|-|\bar{G}_{mn}|]^2 \tag{24}$$

takes the high value $J=23.5932$ for 148 sampling points.

Table 1

Coefficients of the transfer function by the method of Chottera and Jullien

$a_1=0.00150$	$b_1=-0.55852$	$b_8=-500.00000$
$a_2=-0.00114$	$b_2=-500.00000$	$b_9=-500.00000$
$a_3=0.00542$	$b_3=503.28602$	$b_{10}=-396.22889$
$a_4=-0.00659$	$b_4=249.37198$	$b_{11}=-0.04285$
$a_5=0.00379$	$b_5=-500.00000$	$b_{12}=-6.71408$
$a_6=0.00012$	$b_6=-1.23134$	$b_{13}=-0.08952$
	$b_7=398.38451$	$b_{14}=-0.59777$

Solving the above problem by the proposed method (with $Q=1$ in (21)), we obtain the coefficients listed in Table 2 while we find that $\xi=1.5735521$ and $\delta_1=356.677$, $\delta_2=205.580$.

Table 2

Coefficients of the transfer function by the proposed method

$a_1=0.09928$	$b_1=2.90469$	$b_8=20.30024$
$a_2=-0.08421$	$b_2=-4.18640$	$b_9=-11.28699$
$a_3=0.23985$	$b_3=5.39825$	$b_{10}=-23.76603$
$a_4=0.04008$	$b_4=-6.69908$	$b_{11}=5.30287$
$a_5=-0.02279$	$b_5=-8.13214$	$b_{12}=-12.34524$
$a_6=-0.06012$	$b_6=-6.84939$	$b_{13}=-11.25750$
	$b_7=11.76315$	$b_{14}=14.98797$

Moreover, cost function (24) for 148 sampling points takes the value $J=0.0209$ and the value of $\max\{\delta_{mn}\}$ is found to be

$$\max\{\delta_{mn}\}=\max\left||\bar{G}_{mn}-\bar{E}_{mn}|^2-|\bar{H}_{mn}|^2\right|=0.079523. \tag{25}$$

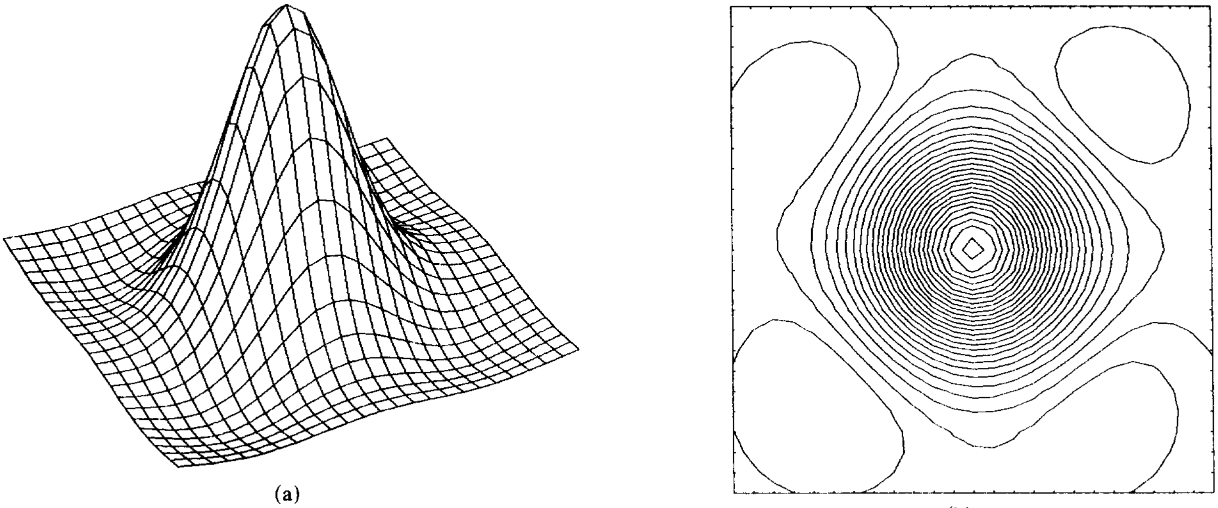
For the sake of comparison, some of the values of the designed magnitude response as well as the corresponding values for a set of sampling points (w_{1m}, w_{2n}) are shown in Table 3.

Table 3

w_{1m}	w_{2n}	Desired values	Values of the designed filter
0.1166	0.06732	1	1.01197
0.17952	0	1	1.00783
0.06969	0.2601	1	1.00319
0.15708	0.27207	1	0.99301
0.10486	0.45942	0.75	0.73935
0.13981	0.61257	0.5	0.49415
0.17477	0.76571	0.25	0.24485
0.20972	0.91885	0	0.00574
0.83012	1.04094	0	0.00496
0.38281	1.67721	0	0.00269
1.80011	2.25726	0	0.00039
2.84828	2.30643	0	0.00068
π	2.56221	0	0.00037

The magnitude frequency characteristic of the desired filter is shown in Fig. 1. Comparing the two methods, the efficiency of the proposed one is obvious. Minimization of the ξ only does not assure an acceptable design for the magnitude response.

Example 2. In this example, a 2-D IIR low-pass filter, whose denominator is a real function, is designed. This means that for the minimization of (16), the satisfaction of constraints (17a) suffices. The transfer



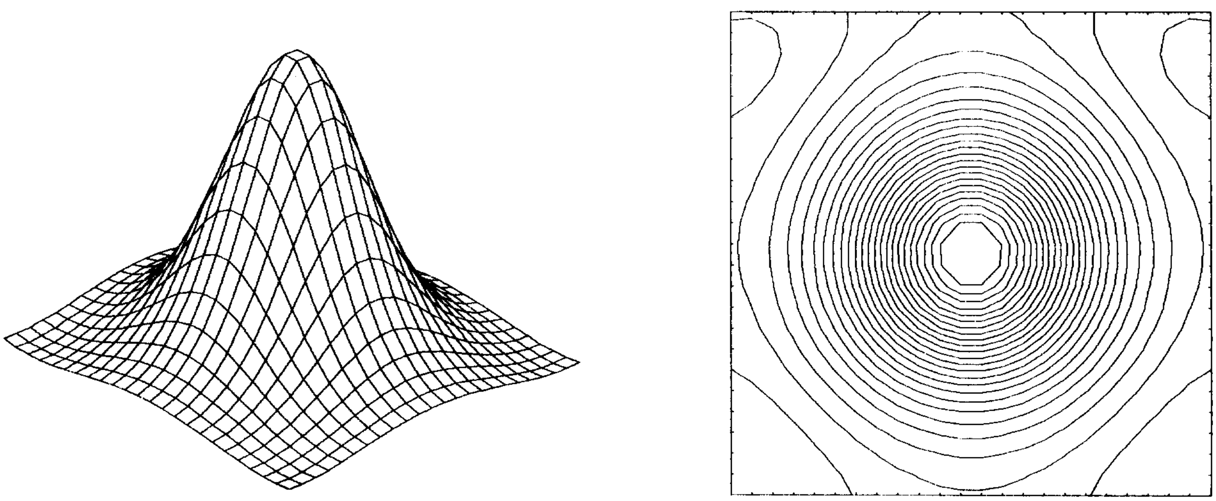
(a) (b)

Fig. 1. The magnitude frequency response of the filter for Example 1: (a) the perspective plot and (b) the contour plots.

function is taken to be of the form

$$\begin{aligned}H(z_1, z_2) = {} & [a_1(z_1^2z_2+z_2+z_1z_2^2+z_1)^4+a_2(z_1z_2+z_1^2+z_2^2+z_1^2z_2^2+1)^4+a_3(z_1^2z_2+z_2+z_1z_2^2+z_1)^2 \\ & \times(z_1z_2+z_1^2+z_2^2+z_1^2z_2^2+1)+4a_4(z_1^2z_2+z_2+z_1z_2^2+z_1)^3(z_1z_2+z_1^2+z_2^2+z_1^2z_2^2+1) \\ & +4a_5(z_1^2z_2+z_2+z_1z_2^2+z_1)(z_1z_2+z_1^2+z_2^2+z_1^2z_2^2+1)^3] \\ & \times[1+b_1+2b_2s_1s_2+b_3(s_1^2+2)(s_2^2-2)+2b_4s_1s_2(s_2^2-3) \\ & +4b_5s_1^2+2b_6s_1s_2(s_1^2+2)+4b_7s_1^2(s_2^2-2)+b_8(s_1^2+2)^2+b_9(s_2^4-4s_2^2+2)]^{-1}, \qquad (26)\end{aligned}$$

where $s_1=z_1+z_1^{-1}$ and $s_2=z_2+z_2^{-1}$. The desired filter specifications have the form of (21) with $G_1=1$, $G_2=0$, $|\bar{E}_{mn}|=0$, $R_p=0.2\pi$, $R_s=0.6\pi$. The total number of sampling points is 156 (56, 28, and 72 in the



(a) (b)

Fig. 2. The magnitude frequency response of the filter for Example 2: (a) the perspective plot and (b) the contour plots.

pass-band, transition-band and stop-band case, respectively). For $T_1 = T_2 = 6$, the coefficients of the transfer function are given in Table 4, while $\xi = 0.02218$ and $\delta = 0.0666$. Moreover, we find that $J = 0.1252$ and $\max\{\delta_{mn}\} = 0.09235$. Note that δ is found to be close to $\max\{\delta_{mn}\}$, as was expected according to (12) for $\mathrm{Im}(\bar{B}_{1m}, z_{2n})) = 0$. The magnitude frequency response of the designed filter is shown in Fig. 2.

Table 4

Coefficients of the transfer function for Example 2

$a_1 = 0.0023985$	$b_1 = 0.0002659$	$b_6 = 0.0116120$
$a_2 = -0.0002922$	$b_2 = -0.1936300$	$b_7 = -0.0005768$
$a_3 = -0.0006459$	$b_3 = 0.0167000$	$b_8 = 0.0002531$
$a_4 = 0.0007517$	$b_4 = 0.0019414$	$b_9 = 0.0121520$
$a_5 = -0.0003963$	$b_5 = 0.0070610$	

3. Conclusions

In this paper, a general method for the design of the magnitude response of a 2-D IIR digital filter by using linear programming is presented. The linear problem is formulated by adding a set of appropriate linear constraints in order to achieve a satisfactory design of the magnitude response. The stability of the resultant filters is always assured by proper selection of the linear constraints. The required computational cost is reduced by exploiting symmetries which appear in the coefficient matrix. Solution of the problem is accomplished in its dual form, by using the revised Simplex algorithm. An additional reduction of the computational cost may be attained by using symmetries in the magnitude response, whenever appropriate. In this case, we select the form of the transfer function so that it possesses some kind of symmetry (e.g., quadrantal or octagonal for the approximation of circular symmetry) by linearizing the terms of its coefficients. The proposed methodology is shown to be a powerful design tool as illustrated by Examples 1 and 2.

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